

# Barn Raisings of Four-Dimensional Polytope Projections

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## Abstract

By constructing large, attractive polytope models, students or a general audience are connected with the beauty of mathematics in a hands-on manner. Four-dimensional geometry provides engaging challenges to think about while working in a team to build a physically impressive, sculptural result. I have led many workshops of this sort with Zometool materials and report on one event in detail. A family of related structures and mathematical ideas that can be taught informally at such an event are also presented.

## 1. Introduction

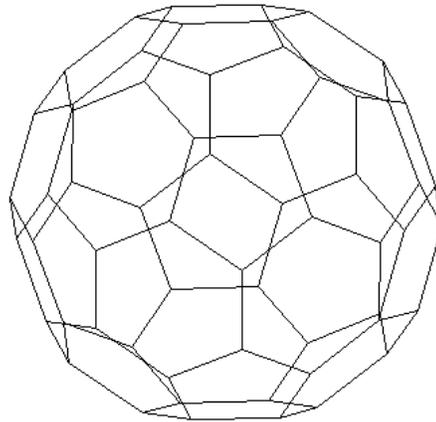
Over the past eight years, I have organized many events where participants work with me to assemble geometric sculptures. I call these events “barn raisings” after the traditional New England events in which a community assembles the framework of a barn. Most people, I believe, have a natural feeling for pattern and form that I call the *geometric aesthetic*. [7] A participatory sculpture activity provides an informal education in mathematics which appeals to this feeling in an entertaining, hands-on manner, attractive even to those who may feel threatened by traditional presentations of mathematics. Examples of various sculpture barn raisings I have led with different designs and materials are described in references [8, 9, 10, 11] and on my web pages. [4]

This paper describes barn raisings in which the structures we build are three-dimensional projections of certain four-dimensional polytopes and the material used is a plastic building set called *Zometool*. [29] Figure 1 shows a close-up of one example, described in detail below in Section 3. It is about 6.5 feet in diameter, constructed from 10800 components. While many polytope models could be made, fifteen relatives of the 120-cell, described below in Section 4, are very natural for the material. They range in complexity from a few hundred parts to over twenty thousand parts, so an appropriate one may be chosen for the number and ability of the participants.

There is a long history—from the ancient Greeks through the European Renaissance to modern art—of incorporating polyhedra in fine art. [3] Twentieth century artists have in some cases been inspired or enriched by knowledge of four-dimensional polytopes. [15] Mathematicians trained to visualize higher dimensional polytopes often find aspects of their symmetry or structure to be beautiful, but find this difficult to communicate to non-mathematicians. So I am happy to be able to provide a direct way for the public to see some of the beauty of 4D objects, in the form of models of their 3D shadows.



**Figure 1.** Author and Zometool polytope model.



**Figure 2.** Truncated icosahedron projected to 2D.

## 2. Polytopes

This section collects mathematical ideas that I present casually in a workshop, as relevant parts of the model are being built. The term *polytope* generalizes polygons and polyhedra to  $n$ -dimensional ( $n$ -D) space. Segments are 1D polytopes. Polygons are 2D polytopes. Polyhedra are 3D polytopes. Mathematicians also study and make models of polytopes from 4D and higher-dimensional spaces. The boundary components of  $n$ -D polytopes are  $(n-1)$ -D polytopes. There are no physical 4D polytopes; we live in a 3D universe and can not construct physical 4D objects. However, it is important to see that ideal 3D geometric objects such as spheres and cubes also do not exist physically in our universe, only approximations to them exist. But this does not stop us from studying their properties. We derive properties of ideal mathematical objects and find ways to model and understand them. Projections provide one way to study ideal 4D objects.

A 3D object held in sunlight casts a 2D shadow, a projection one dimension lower. Figure 2 illustrates this with a shadow of the edges of a truncated icosahedron, better known as the form of a soccer ball (in the US) or football (in Europe). Our visual systems can deal with two artifacts of this projection: (1) foreshortening causes the regular pentagons and hexagons of the 3D original to appear compressed in 2D; and (2) the front and back of the shadow overlap, giving a kind of “Necker cube effect”, e.g., we might see either of the two central pentagons in Figure 2 as the front. Except for the outer rim, all of Figure 2 is “double covered”. Analogously, we can calculate the 3D object that is the one-dimension-lower shadow of an envisioned four-dimensional object. The two types of artifacts mentioned above must be considered when interpreting these projections: (1) foreshortening results in 3D parts that appear compressed in 3D even though they derive from regular components in 4D; and (2) for the polytope projections considered here, there will be no “Necker cube effect” because the front and back shadows exactly overlap. But we must understand each part of the projection (except the outermost rim) to be a double coverage, representing two distinct parts of the original.

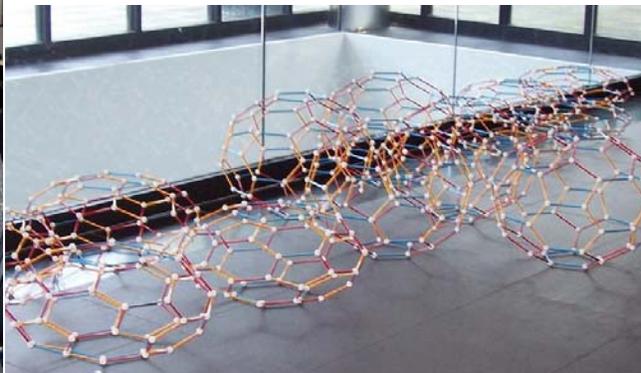
The truncated icosahedron implied by Figure 2 is an example of a “uniform polyhedron,” meaning each face is regular and each vertex is equivalent. At each vertex of a soccer ball, one regular pentagon and two regular hexagons meet. The term “Archimedean” is sometimes used to describe such polyhedra. The 120-cell is a uniform 4D polytope consisting of 120 regular dodecahedra, four at each vertex, three around each edge. It is the simplest member of a family of fifteen uniform 4D polytopes related to each other by “expansion” (separating components) and/or “truncation” (slicing off vertices). Alicia Boole Stott, in the early twentieth century, was the first to mathematically describe many members of the 120-cell family. [22, 24] Various people have made polytopes projection models using e.g., cardboard, or wire

and thread [13, 19] or computer graphics [19, 27] or solid freeform fabrication [12]. More recently, it has been possible to construct even the most complex models from this family using Zometool.

Zometool was designed around 1970 by the architect Steve Baer for purposes such as dome models. However, it incorporates lengths and angles that are useful for many other structures. [14] The parts fit together quickly and easily by friction, with no tools. So they are suitable for group construction projects with participants of all ages. I have found them ideal for polytope barn raisings.



**Figure 3.** *Start of Construction*



**Figure 4.** *Truncated Icosahedron Modules*

### **3. A Modern-Day Sangaku at Stony Brook University**

The polytope construction event described in this section took place at Stony Brook University, April 24, 2006. With announcements to the entire campus community, I promoted it as an interdisciplinary activity and exhibit, blending art, mathematics, computer science, and an aspect of traditional Asian culture. *Sangaku* (also written *san gaku*) is a Japanese tradition of celebrating geometrical beauty, popular during the Edo period (1603-1867) by members of all social classes. The traditional sangaku is a shaped wooden tablet on which a geometric problem or theorem is written with colorful paint. The tablet was displayed in a temple or shrine as an offering, as a puzzle for others to think over and understand, and as a way for the creator to show off the geometric discovery. Many are so beautiful that they are considered works of art. Although most sangaku are lost, almost nine hundred survive, scattered all throughout Japan. [23]

I proposed to create a modern-day celebration of geometric beauty, in the tradition of sangaku, by leading members of the Stony Brook community in a large colorful geometric construction. Our temple or shrine-like environment for displaying the result was the Charles Wang center, which provides a serene atmosphere for people to contemplate the beauty of the geometric form. Although we updated to a three-dimensional construction from the traditional two-dimensional tablet, its content, like many sangaku, contains numerous nested and tangent circles and spheres.

In Figure 3, participants are assembling various modules at the start of construction of a “Cantitruncated 600-cell”. As far as I know, this was a world-premiere event. Some of the required 10800 Zometool parts are on the tables. The complete structure requires 3660 nodes and 7140 struts. Specifically, 1440 short red, 1440, medium red, 2400 medium yellow, and 1860 medium blue struts are needed. Parts counts for planning this and related polytope models are discussed below.

The components of a 3D polyhedron are 2D faces, e.g., a truncated icosahedron consists of twelve regular pentagons and twenty regular hexagons, which meet at edges. Analogously, one dimension up, a 4D

polytope consists of 3D “cells” which meet at their faces. Our polytope contains three types of cells: 120 truncated icosahedra (TI), 600 truncated octahedra (TO), and 720 pentagonal prisms (PP). Various sequences are possible for constructing and assembling these cells. The plan I chose was to build the TI cells individually and join them with struts that outline the PPs. Doing this, the TOs are automatically formed in the spaces between the hexagons of neighboring TIs.

Figure 4 shows some of the TI cells ready to be assembled together after an hour or two of construction. There are 75 compressed soccer balls in five layers (1+12+20+12+30) from center to exterior. Different edge lengths and angles correspond to different degrees of foreshortening. Teams of participants build first one type of cell then another, learning the different types of cells and how they are related. Artists are familiar with the notion of foreshortening, e.g., a circle seen from an oblique angle is drawn on canvas as an ellipse, in accordance with the geometry of perspective. This transformation occurs when we project a 3D reality on to the restricted 2D space of the canvas. However, it takes some hands on experience to understand that in 4D, all the cells of our polytope are quite symmetric, although they appear progressively flattened in 3D as we move away from the center, with the outer ones totally flat.



**Figure 5. Partial Construction.**



**Figure 6. Reaching the Top!**

Figure 5 shows the center module, a blue TI at the top center of the partial construction. We started with the center and worked our way toward the bottom. When complete, the center TI will be surrounded on twelve sides with a layer of slightly foreshortened TIs. Five struts are used to connect pentagon to pentagon, which creates cells in the form of PPs. Then additional layers of TIs surround these, each successively more foreshortened.

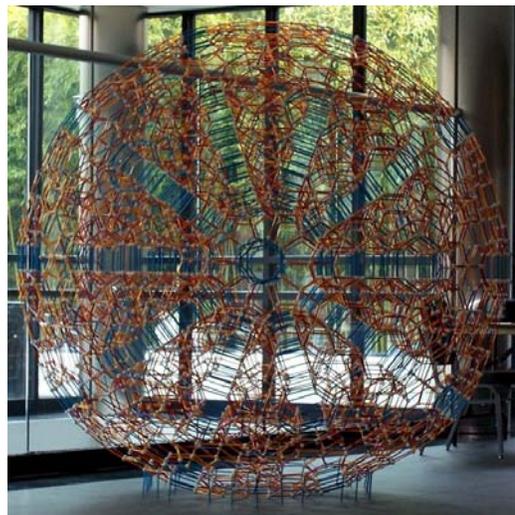
A number of engineering considerations must be addressed during construction. Because we are making a large spherical form, 6.5 feet in diameter, it would be difficult to reach the top center if we saved it for last. So we first built a tall central core straight up to the top. This is shown in Figure 6. After the central core is formed, many people can work on it from all sides. Cells are added to the central core in pairs, always having two teams work simultaneously from opposite sides, so the structure can not become unbalanced and tip over. For strength, some vertical struts are added to carry the load, even though they are not part of the mathematical form. It would be great if in the future Zome manufactured clear parts to use for this purpose.

Figure 7 shows the construction after 3.5 hours. We are attaching the outer layer of thirty cells that are completely foreshortened to two-dimensional flatness. Even before this point, I could walk away and let

the group work out its own path to completion. During construction, I give specific mechanical instructions like “connect part A to part B” and a casual presentation of the mathematical ideas from Section 2. Each participant comes to their own understanding of the geometry, e.g., seeing it is uniform because four polyhedra meet at every vertex: a TI, a PP, and two TOs. I am always gratified to get to the point when some of the participants start answering the questions of others. When they can explain the process to each other and together solve the problems of how to attach the next cells, I know I have done my job as a teacher.



**Figure 7.** Nearly Finished.



**Figure 8.** Completed Polytope Model.

Figure 8 shows the final result, after four hours, viewed along a 5-fold axis. It doesn't have any common name. The name used in Norman Johnson's upcoming book on uniform polytopes is the *Cantitruncated 600-cell*. [16] One may also call it the *Truncated Ambo 600-Cell* as that may help in visualizing it. The projection model is impressive visually and for the amount of loving labor involved. Everyone finds it colorful and attractive and wants to have their photo taken with it. The views along its various symmetry axes are particularly impressive. In addition, I think that many people find these events fun because they are an intellectual challenge that presents stimulating new patterns and relationships to think about.

#### 4. The Fifteen Related $H_4$ Polytopes

The above construction illustrates just one of the fifteen convex uniform 4D polytopes that can be built with Zometool, each projected so the front and back overlap. They share the same symmetry group, the Coxeter group,  $H_4$ , of order 14400. [2, 20, 28] Table 1 summarizes this set of polytopes, many of which I have built at various workshops at universities or conferences [1, 21], e.g., as part of a geometry exhibit at the Chiang Kai-Shek Memorial in Taipei, Taiwan. At the *Matematica e Cultura* conference in Venice, 2006, I was invited by Michele Emmer to do an outdoor public construction. This workshop took place in a lively city square in Venice, the Campo Sant' Angelo. Given the uncertainties of the environment, I chose a can't-fail polytope that is very easy to parallelize: the expanded (or “runcated”) 120-cell. Participants from the conference and curious members of the public worked together to assemble 4860 parts into a structure over a meter in diameter, shown in Figure 9. [5] Because of its very light and open form, my favorite of the fifteen is the truncated 120-cell. This was first constructed at a workshop I led at the Canada/USA MathCamp in UBC in Vancouver, in 2000. [17] Figure 10 shows one at a more recent construction I led at Dowling College, NY, in 2006. [6] It is rather fragile and requires a stable environment and cautious workers. Reference [28] gives a very readable illustrated introduction to the mathematical ideas underlying these polytopes.

Many other people have also made some of these models and/or organized polytope construction events, including Marc Pelletier, Steve Rogers, Paul Hildebrandt, Mira Bernstein, Vin de Silva, David Richter, and Dan Duddy. A web page with links to photos of some of these constructions is maintained by Dave Richter. [26] Recently, the larger models have been possible, due to Zome's redesign for tighter connections and a shorter length of parts. The largest of the fifteen has 21360 parts. It was first constructed at the London Bridges 2006 Conference [1] in all-day event organized by Hildebrandt, Duddy, and Richter. We expect the model will be displayed at the London Science Museum.

For others who may wish to organize such events, I offer the following suggestions. First, become familiar with Zometool and make the 120-cell. Instructions for it are available in [14] and in a special 120-cell kit which Zome sells. The truncated 120-cell is also outlined in [14]. When making the 120-cell, observe there are four important types of location. The fifteen  $H_4$  polytopes differ in terms of what sort of cell, if any, appears at each type of location. Name these locations I, T, P5, and P3 as follows: The centers of the dodecahedra are locations of icosahedral symmetry (I). The vertices of the 120-cell are locations of tetrahedral symmetry (T). The pentagon face centers are locations of pentagonal prism symmetry (P5). The midpoints of the struts are locations of triangular prism symmetry (P3). Table 1 shows which sort of cell to place at each location in each polytope.

All fifteen polytopes can be made with the same four types of struts: short red, medium red, medium yellow, and medium blue. The parts count for each is given in Table 1, as computed by Richter. [25] There is always the same number of the two red lengths, so only one column is given for reds. The strut lengths can be scaled down to Zome's new smaller sizes for more rigid models.

Ultimately, it would be magnificent to build all fifteen as an exhibit at a single location, so one can directly see the connections between them. If this is done, I propose that the fifteen be arranged in space in a giant tetrahedron formation, which would be a kind of "periodic table" of their relationships. Coxeter provides a construction for these polytopes in terms of a vertex lying on zero, one, two, or three of the four mirror planes that generate the group. [2] The four vertices of the tetrahedron formation correspond to the four mirror planes. The four polytopes that are generated by the vertex off just one mirror plane would be at the corners of the tetrahedron. The six polytopes that correspond to the vertex lying off two mirrors would be at the corresponding edge midpoints. The four polytopes that correspond to the vertex lying off three mirrors would be at the tetrahedron's face centers. And the remaining and most complex polytope, in which the vertex lies off all four mirrors, would command the tetrahedron's center.



**Figure 9.** *A Polytope in Venice*



**Figure 10.** *Truncated 120-Cell*

Each  $H_4$  polytope has 14400 symmetries, with a fundamental domain (replicated 14400 times) bounded by four hyper-planes which are mirror symmetry planes. [2] If we place a vertex in this fundamental domain, we obtain the vertices of the "Omnitruncated 120-cell" (line 1111 of Table 1). If we place that

vertex on one of the boundary planes of the fundamental domain, pairs of mirrored vertices coincide and we get only 7200 vertices. If that generating vertex falls simultaneously on more than one mirror plane, we get a larger reduction in the number of vertices. In four extreme cases, when the vertex lies on three planes, we obtain the simplest cases, e.g., the 600-cell (line 0001 of Table 1). The Wythoff symbol has four bits of information, corresponding to the four generating mirror planes. The entry 0 or 1 corresponds to the vertex lying on or off the corresponding mirror. The I, T, P5, and P3 columns list which type of cell appears at each location, with cell types abbreviated as: T=tetrahedron, O=octahedron, CO=cuboctahedron, I=icosahedron, D=dodecahedron, ID=icosidodecahedron, rID=rhombicosidodecahedron, t=truncated, and  $P_n=n$ -gon prism. In 4D, there are 120 of whatever cell appears in the I column, 600 of the cell appearing in the T column, 720 of the cell in the P5 column, and 1200 of the cell in the P3 column. But in the 3D projection, because the front and back overlap there are only seventy-five type I cells to build. Table 1 includes Norman Johnson's names. [16]

## 5. Conclusions

The lengths and angles built into Zometool make it relatively easy to build beautiful, large models of 4D polytopes from the  $H_4$  family. For several years, I have been going around the world leading polytope barn raisings. One event, presented as a modern-day sangaku, was described in some detail. Others are similar but with different modules. In the process of construction, there are many opportunities to introduce the underlying mathematical ideas casually. These hands-on activities introduce mathematical ideas in an informal non-threatening way. The interpersonal dynamics of a large group of people working over an extended time towards a challenging goal results in a different type of learning experience than most people associate with mathematics. Group activities such as this are one way to move mathematics in the stream of popular culture. In addition to the colorful attractiveness of the 3D model, participants appreciate abstract ideas of higher-dimensional geometry, which hold their own form beauty.

## Acknowledgements

Thank you to everyone who participated in all my barn raisings and to Zometool for providing the parts for many of these constructions. This paper benefits from discussions with many people who have participated in workshops with me over the years.

Wythoff	I	T	P5	P3	Parts	Balls	Reds	Yellow	Blue	Name
0001	.	T	.	.	351	75	72	120	120	600-cell
0010	I	O	.	.	2196	396	360	600	480	Rectified 600-cell
0011	I	tT	.	.	2964	780	432	720	600	Truncated 600-cell
0100	ID	T	.	.	2440	640	360	600	480	Rectified 120-cell
0101	ID	CO	P5	.	7200	1860	1080	1800	1380	Cantellated 600-cell
0110	tI	tT	.	.	5460	1860	720	1200	960	Bitruncated 120-cell
0111	tI	tO	P5	.	10800	3660	1440	2400	1860	Cantitruncated 600-cell
1000	D	.	.	.	950	330	120	200	180	120-cell
1001	D	T	P5	P3	4860	1260	720	1200	960	Runcinated 120-cell
1010	rID	O	.	P3	7200	1860	1080	1800	1380	Cantellated 120-cell
1011	rID	tT	P5	P6	12540	3660	1800	3000	2280	Runcitruncated 600-cell
1100	tD	T	.	.	3680	1260	480	800	660	Truncated 120-cell
1101	tD	CO	P10	P3	12540	3660	1800	3000	2280	Runcitruncated 120-cell
1110	tID	tT	.	P3	10800	3660	1440	2400	1860	Cantitruncated 120-cell
1111	tID	tO	P10	P6	21360	7200	2880	4800	3600	Omnitruncated 120-cell

**Table 1.** Data for Fifteen  $H_4$  Polytopes

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