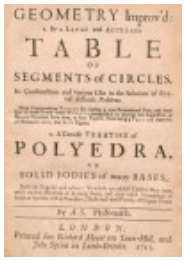
The Multifaceted Abraham Sharp

George W. Hart

1. Introduction

Abraham Sharp (1653?-1742) was a mathematician and scientific instrument maker who worked for several years as assistant to England's first royal astronomer, John Flamsteed, receiving great praise for the high quality of his workmanship. But equally noteworthy are the accomplishments that went into his remarkable 1717 book, *Geometry Improv'd* [19]. Figure 1 shows the title page. It is actually two independent books packaged under one cover. We will ignore the first 64 pages, which concern trigonometry tables, and focus on Part 2 of the book, *A Concise TREATISE of POLYEDRA or SOLID BODIES of many BASES*. Sharp presented twelve original "solid bodies" and detailed a unique method by which they could be constructed. These range in complexity from 18 to 120 faces and include examples that were later rediscovered. Although Sharp's polyhedra are significant for being new to the geometry literature, they have been almost entirely ignored for three centuries by the mathematics community. *Geometry Improv'd* is unusual in many ways and I maintain that the only way to make sense of its contents, its style of presentation, and its poor reception, is to think of Abraham Sharp as an artist.


GEOMETRY Improv'd

TABLE

OF

SEGMENTS of CIRCLES.

POLYEDRA.

SOLID BODIES of many BASES.

Figure 1. Title page of *Geometry Improv'd*

There has never been another mathematics text that compares to the obsessive labor of love in which Sharp meticulously presents his twelve original polyhedra. What stands out most saliently is how he gives all the necessary dimensions both as exact formulas (with "surds," i.e., irrationals expressed by radical signs) and as overly precise numerical values. His decimal numbers are painstakingly hand-calculated with sixteen to thirty digits of precision, sometimes via intermediate terms up to fifty digits long. The book often presents multiple detailed constructions starting variously from a cube, a rectangular parallelepiped of specified proportions, a sphere, and/or a previously prepared simpler body (dodecahedron, icosahedron, or rhombic triacontahedron) to attain the same intricate result. Furthermore, redundant alternate ways to locate the same point are often included, e.g., the distance to measure in from a corner and the distance to measure out from a center are sometimes both stated.

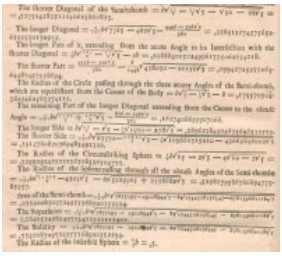


Figure 2. Sharp's text showing exact values and decimal approximations (up to 28 digits) of dimensions, surface area, and volume of one of his original solids.

Figure 2 shows a typical section of Sharp's text. A scanned copy of the full book is available online [19]. The concise treatise of polyhedra is packed into only 32 pages, but one immediately gets the impression that Sharp may have put years of passionate work into the calculations alone. He also made wooden models that must have required a great deal of time. Although they reify his methods and validate some of his numbers, they are not mentioned in the book.

Sharp made three large copperplate engravings, “neatly engraved by his own hands” [15], that illustrate *Geometry Improv'd.* These are works of art in their own right and, like the text, they are overwhelming in their density of information. Figure 3 shows his Plate 1, illustrating just the simpler forms from the book. This one page includes eighteen examples of how to start with a solid block (a cube or a parallelepiped) and mark the surface to construct slicing planes that pass at appropriate angles through the interior. After cutting off all the exterior material (beyond the slicing planes) the shape remaining at the center of the block is the desired polyhedron.

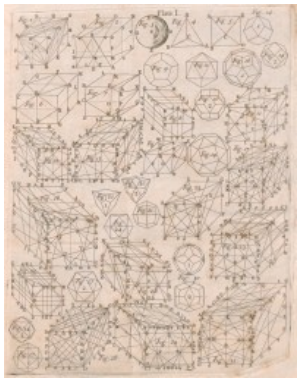


Figure 3. Plate I of *Geometry Improv'd*

In Figure 3, Sharp's Plate I, Fig. 16, parts 1 and 2 show two views of the “very elegant” rhombic triacontahedron, which comprises thirty congruent golden rhombi. His Fig. 17 indicates where the surface of a cube is intersected by the 24 slicing planes needed to release the polyhedron from the interior of the cube. (Each of the 24 slices produces one face, plus six faces arise as the central uncut portions of the cube's original square faces, to make the total of thirty rhombic faces.) He notes that it is “so commodiously cut from a Cube as to exclude all other Methods.” This “subtractive” method is a very different way to make polyhedra than the tape-together-paper-faces technique which is now standard for introducing students to new 3D forms. It evokes Michelangelo working with his chisel more than the Euclidean construction of Platonic solids in *The Elements*. Figure 4 is my reconstruction—a rhombic triacontahedron cut from a solid block of wood by following Sharp's diagrams and using his calculations.

Figure 4. Wood Rhombic Triacontahedron, sliced by the author

Sharp's unusual methods and laborious presentation undoubtedly contributed to his twelve original polyhedra being long ignored. It is quite an ordeal to carefully read his book and make sense of his geometric intentions. From the dearth of citations to his work in the mathematics literature, I have come to doubt that anyone in the past 300 years other than Sharp and me has ever thoroughly read it all.

At least two of Sharp's forms were later rediscovered by others: He describes a body with 24 kite-shaped faces that is now called the “deltoidal icositetrahedron.” It is an example of a “Catalan Solid,” named after the 1863 paper in which Eugène Catalan described it, the dual to the rhombicuboctahedron, 150 years after Sharp did [4].

And Sharp's body with 60 isosceles triangular faces is exactly what Buckminster Fuller called a “Class II geodesic sphere,” an elevated dodecahedron with all vertices the same distance from the center, so that it is inscribable in a sphere.

While the book is certainly troublesome to read, I believe *Geometry Improv'd* has been ignored mostly because Sharp does not discuss the original construction ideas underlying the novel forms. This is not the type of math book that aims to communicate insightful understanding of any underlying structure, nor to explain the relationships between the new discoveries and previous work, nor to teach general methods that might be used in solving future problems. While ostensibly aimed towards a mathematics audience and claiming that the calculations provide “Exercise and Improvement of the Doctrine of Surds,” the book does not present the style of organized thought that a mathematician looks for. It is filled mainly with the tedious minutiae of how to reproduce particular examples. The results can only be appreciated after following the laborious steps and attempting to re-imagine the designer's creative process. Sharp also did not explain why one should follow his rules other than to claim that the results will be “Elegant and Beautiful”.

The remainder of this paper is organized as follows: First some background is presented about Abraham Sharp and his unusual book, to place it in historical context. Then the wooden models he created are described, along with the little I have been able to find out about what happened to them after his death. Next, I consider in detail just one of Sharp's twelve original forms, one so notable that I informally call it “the Sharpohedron”. My intention is to explain some of what is left out from the book: the mathematical thinking that enriches our understanding of the form and allows us to interpret it in a wider context of geometric knowledge. No doubt some of these broader ideas inspired Sharp's creative process, but we can only speculate. The concluding section is a more subjective exploration of how Sharp's approach to his work has an organic coherence when he is viewed as an artist.

2. Abraham Sharp's Life and Work

> *Twelve* (I presume) *New Geometrical Bodies* are herein to be exhibited to publick Notice, some of which may possibly be as Elegant and Beautiful (if truly form'd according to the Rules herein prescrib'd) as any of that Nature hitherto known.

With these words from *Geometry Improv'd,* Abraham Sharp introduced his twelve novel polyhedra. That apostrophe-D participle form was common and *X Improv'd* was a meme of sorts in the bookstalls of its time. An online search turns up scores of similar British book titles from the late 1600s and early 1700s: *Trigonometry Improv'd, Surveying Improv'd, Farriery Improv'd, Hocus-Pocus Improv'd,* etc. The Enlightenment was underway and a reading audience sought new knowledge.

The Platonic solids were well known from Euclid's *Elements,* and an assortment of other symmetric polyhedra had sporadically appeared in print. As prelude to his twelve original bodies, Sharp first describes how to cut out from blocks seven solids that had been previously described in the mathematics literature: the tetrahedron, octahedron, dodecahedron, icosahedron, rhombic dodecahedron (called simply “the Body of Twelve Rhombs”), the rhombic triacontahedron mentioned above (called “the Body of Thirty Rhombs”), and the snub cube (discussed below).[1]

Though these seven polyhedra existed in the literature of the time, Sharp's method of cutting them from solid blocks is entirely original. I know of no other mathematics reference, before or since, that presents polyhedra by his technique of calculated slicing. The closest analog is in the ornamental woodworking tradition. For example, the nineteenth century Holtzapffel reference volumes give the compound miter angles to cut simple polyhedra from blocks and describe turning simple polyhedra on a lathe in a spherical chuck, starting from a solid wood or ivory sphere[2] [13]. From the way Sharp describes first marking the surface of the block to identify the cutting plane and then making cuts along the lines, one assumes this is to be done with a hand-saw. But physical processes and materials are never mentioned, only the abstract geometrical steps.

It is not known how Sharp came to be in a position to make scientific instruments or precision polyhedra. He was born in Little Horton, near Bradford, in Yorkshire, but much of his life is undocumented, starting with his year of birth. A valuable early reference, the entry on Sharp in Charles Hutton's 1795 *Philosophical and Mathematical Dictionary,* says "he was born about the year 1651" [15]. But in 1889, almost 150 years after Sharp's death, when a trove of his correspondence was discovered, William Cudworth wrote the only book-length biography of Sharp and claimed he was born in 1653 [6, p. 200]. Recent scholarship is more cautious, e.g., the *Oxford Dictionary of National Biography* only states his documented baptism year of 1653 [23].

Sharp learned his Latin and mathematics at a local grammar school in Bradford, before being apprenticed to a wool merchant in York at the age of 16. But that did not work out. Cudworth states that Sharp "did not take kindly to the yardstick and counter" and that his interests "unfitted him for dealing in dimity and calicoes." He abandoned his apprenticeship and fled to Liverpool where "he opened a day school and taught writing and accounts" and met the astronomer John Flamsteed. Before long he was working for Flamsteed and in that capacity he created one of the most celebrated astronomical instruments of its time, the Mural Arc at the Royal Observatory, Greenwich [5].

Flamsteed later wrote: "The making of this [instrument] was principally the work of Abraham Sharp, my most trusty assistant, a man enriched with gifts and resources of every kind to render him competent to complete a work so intricate and difficult" [6]. Elsewhere, he calls Sharp "a man much experienced in mechanics, and equally skilled in mathematics." Hutton gushed: "Indeed few or none of the mathematical instrument-makers could exceed him in exactly graduating or neatly engraving any mathematical or astronomical instrument, as may be seen in [the Mural Arc], or in his sextant, quadrants and dials of various kinds; also in his curious armillary sphere, …, his double sector, with many other instruments, all contrived, graduated, and finished, in a most elegant manner by himself." How exactly Sharp went from the wool counter to world-class expertise in instrument-making and the art of engraving is a mystery. Much of his knowledge appears to have been self-taught.

After completing the Mural Arc in 1690, Sharp left Flamsteed's employ. His occupations are not certain for a few years, but likely included teaching and work as an instrument maker. Then Sharp's eldest brother died in 1693 and he moved to the family estate at Little Horton in 1694 to manage it for his widowed sister-in-law. When his nephew died in 1704, Abraham Sharp became heir to the estate and apparently enjoyed the gentlemanly leisure to work on whatever pet projects he wished. He spent the rest of his life as a wealthy recluse at Little Horton, unmarried, traveling hardly at all and receiving very few visitors, but connected to the scholarly scientific world by correspondence.

An undated portrait of Abraham Sharp (by an anonymous painter) on display at Bolling Hall, Bradford, shows him in gentleman's clothing, holding some of his instruments [1]. In 1744, two years after his death, this portrait was the model for a printed engraving of Sharp by the prolific engraver George Vertue. See Figure 5. In this posthumous engraving, a frame-within-a-frame motif places Sharp in a separate realm from his earthly instruments and a rectangular panel is portrayed that includes a geometrical figure (discussed below). A detailed study of this portrait [20] notes that Vertue sold themed collections of engravings, e.g., of poets or royal houses, and this portrait of Sharp "represents perhaps the earliest attempt by members of the popular culture to 'possess' collections of intelligent and powerful people by buying posters."

Figure 5. Engraved Portrait of Abraham Sharp by George Vertue

As a gentleman of leisure, Sharp continued to make instruments and chose to occupy himself at length with meticulous calculations. He carried out his computational work for no fee, simply for his own entertainment. Sharp volunteered as an astronomical computer for Flamsteed, calculating predicted positions for the planets, the moon, and Jupiter's satellites, plus tables of data that eventually appeared in Flamsteed's posthumous *Historia Coelestis Britannica*. In 1699, he worked out "for his own amusement" (using 150 terms of an infinite series for the arctangent function) the decimal value of π to 71 digits, doubling the world record of the time. He did this in two different ways so he could check for correctness. He also produced a table of 60-digit logarithms, published in 1705 by Henry Sherwin. These are impressively difficult accomplishments, but in addition, at some time in this period he also produced the original polyhedral designs, twenty-plus digit calculations, detailed engravings, and wood models of *Geometry Improv'd.*

What motivated Sharp in his polyhedra project? He suggested no practical application for his twelve new bodies.[3] He received no payment for his book from the publisher, only a dozen printed copies. He did not even seek authorial fame, modestly publishing under the pseudonym "A. S. Philomath." He certainly understood that twenty digits of accuracy gave far more precision than any physical construction could ever require.[4] And yet he put an enormous amount of time and work into this project. Sharp gives no indication of intent other than a desire for beauty and elegance. His twelve solid bodies and their peculiar presentation are a gift to the world from an obsessive, passionate creator. For me, they are art.

Apparently the world was not ready for this type of art, as *Geometry Improv'd* has been little noticed in the ensuing three centuries. Some copies must have sold as there was both a 1717 and a 1718 printing, but I conjecture that the demand was mostly for the practical tables in the first half of the book. There are almost no citations of the geometry portion of the book and it had zero impact on the subsequent development of polyhedra theory.[5] But I don't think that would have bothered him. Sharp was a truly modest and generous man, known for a peculiar habit in donating to the poor. He would walk through town every week with his hands behind his back holding a pile of coins. People in need could come up behind him and pluck money from his hands without having to reveal themselves. I see a parallel in his artistic generosity: we all are invited to follow in his footsteps and pick out what we wish from what his book offers.

3. Sharp's Boxwood Polyhedra Models

Sharp's presentation in *Geometry Improv'd* is purely geometric. We are instructed to cut off prisms, pyramids, or segments from a cube, but never to saw a chunk from a block of wood or other concrete material. There is no mention at all of physical cutting tools such as saws. In this purely abstract realm, the book makes no mention that Sharp actually made wooden models. When I first read it twenty-five years ago, I wrote a web page emphasizing how his thinking was like that of an artist [10]. But I didn't know then that he had also made wooden models of his twelve solid bodies, or I might have been tempted to also describe him more specifically as something of a sculptor. I was surprised years later to happen upon Hutton's summary of Sharp's life and to read: "...the models of these polyhedra he cut out in boxwood with amazing neatness and accuracy" [15].

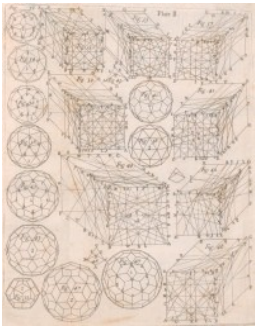


Figure 6. Plate II of *Geometry Improv'd*

For Sharp to make accurate wood models of his polyhedra was an impressive accomplishment. His more complex bodies have 60, 80, 84, 90, and 120 faces, as can be seen in his Plate II, Figure 6. It is one thing to calculate precise dimensions on paper, but quite another to be true to them in the physical world. Why did he not mention that accomplishment in his book? This might be another example of his modesty, or perhaps Sharp wanted to position his discoveries in a lofty mathematical realm and not debase them with Earthly matters? I wish he had written at least a little about the woodworking process so we would have more specific information about his methods. I have reproduced all twelve of his original polyhedra by 3D-printing, in which the machine is responsible for the fabrication accuracy. But I have made only a few from solid wood blocks by his slicing method, which demands considerable time and care to obtain an exact result.

If Sharp had written about the making of his wooden polyhedra, perhaps they might have been recognized as an important part of his work and so better preserved. I have contacted many museums and historical societies trying to track them down, but with little success so far, mainly finding tantalizing records in a few old collections, since dispersed. His twelve original bodies are not conserved in any major museum as they deserve, but I still hold some hope that they may be sitting as under-appreciated curios on a dusty shelf in some British manor house, waiting for an informed viewer to recognize them.

The earliest evidence I have found of the wood polyhedra is in a letter from Sharp to Flamsteed, dated Feb. 2, 1701-2 [9]. Their correspondence generally concerns astronomical matters, but in this letter Sharp mentions: “I have some time agoe made 12 new Geometricall Bodys, ... some of these you may possibly have seen or heard of per Mr Kirk when he was at London about 2 years agoe, whither at his desire I sent 3 of them to him.” On Feb. 6, Flamsteed, who was a Fellow of the Royal Society, responded “I heard your bodyes were before the society but saw them not.” It is unclear if Flamsteed is saying the polyhedra were physically displayed at a Royal Society meeting, or if someone gave a presentation talking about them, or what precisely. I have not been able to locate any more concrete information, but this suggests that additional documentation about the wood models may be preserved somewhere in the minutes of a Royal Society meeting or in notes or letters of someone who attended.

As to the three models that Sharp says he gave “Mr Kirk,” they seem to have have left more of a trail. This was Thomas Kirke, FRS, (1650 – 1706) who is occasionally mentioned in Sharp's letters and Cudworth's biography. Seven years after Kirke's death, a 1713 catalog of the *Musaeum Thoresbyanum* lists material both from Sharp and from Kirke [21, pp. 489 and 498]. This musaeum was a substantial “cabinet of curiosities” rich with relics and marvels assembled by the celebrated antiquarian Ralph Thoresby, who was also known to Sharp. Three of the musaeum catalog entries are intriguing, though frustrating:

- A Body of thirty Rhombs composed by the late ingenious Virtuoso Tho. Kirk Esq; F.R.S.
- Other larger Mathematical Bodies.
- Some Mathematical Bodies by the curious Pen of the incomparable Mr. Sharp

Was the listed rhombic triacontahedron truly *by* Kirk, or was it perhaps received *via* Kirk and actually one of the models that Sharp mentions he gave to Kirk? What were the other large bodies and were they by Sharp? And should we interpret “bodies by Sharp's pen” to mean drawings or are they wood models that he first designed? We can only speculate about these questions as Thoresby's collection was dispersed at auction after his death. The 1764 London sales catalog lists many antiquarian items, but no geometrical bodies [3]. I have found no record of what might have happened to them.

Cudworth's biography contains a three-page inventory of Sharp's Instruments that was “probably made after his death.” It includes telescopes, quadrants, micrometers, sectors, a microscope, sundials, a large burning glass, a fine large lodestone, a quantity of boxwood, and much else that one might interpret in the act of gauging a man's life by what remains after his death. One specific entry is very encouraging: “Curious Set of Solid Bodies No 24”. I expect that these include the twelve original boxwood polyhedra mentioned by Hutton. Unfortunately, these twenty-four curious models, along with almost everything else Sharp made or owned, are not to be found. Investigating in the 1880s, Cudworth reported that “Many of the instruments with historic associations have

been utterly lost, and the materials of others have had narrow escapes of being put to debased uses. A few only are known to be in safe keeping."

Before Cudworth, the Reverend N.S. Heineken had made a search in the 1840s for any "Relics of the Mechanical Productions of Abraham Sharp" [12]. He tells the sad tale of what happened to many of Sharp's hand-written papers after his death: "many years since, when they had been neglected by the owner of the house and left in a closet, the cook was in the habit of supplying herself from the ample store for the purpose of lighting fires and singeing fowls!" It is painful to imagine Sharp's many pages of hand calculations and geometric diagrams being reduced to ashes in this way. More positively, Heineken is able to report on the location of a few instruments and states: "some geometrical solids, turned in the first-mentioned lathe, now belong to my friend J. Waterhouse, Esq., of Well Head near Halifax." I have found that some items from Waterhouse's collection were passed on to the Halifax Literary and Philosophical Society Museum, and from there some went on to other area museums, but I have found no record of the polyhedra John Waterhouse once held.

Among all those dead ends, I can report one positive result. Three of Abraham Sharp's wooden models are preserved in the collection of Bolling Hall of the Bradford Museums and Galleries. The only record of their provenance is that they came to the museum in 1916, with the donor recorded as F.S. Bardsley-Powell. This is Sir Francis Sharp Powell, a descendant of the Sharp family, baronet, and member of parliament [14]. He died in 1911, so the donation most likely came from his wife, Lady Powell. The Powells resided in the Little Horton Hall where Abraham Sharp had lived and worked most of his life, so it seems likely these three wooden models simply sat there for 200 years. Might there be others? The building was demolished in the 1960s.

Figure 7. Three wood polyhedra made by Abraham Sharp: rhombicuboctahedron, icosidodecahedron, and snub cube. Image courtesy of Bradford Museums and Galleries.

Figure 7 shows the three wooden solid bodies made by Sharp, held in the Bolling Hall museum: a rhombicuboctahedron, an icosidodecahedron, and a snub cube. All three are Archimedean solids that appear in the literature predating him, so these are not any of his twelve original forms. The first two are not mentioned in *Geometry Improv'd*, but could have been included among the simple introductory examples. The first is easy to cut from a cube and the second is trivial to cut after first following Sharp's instructions to make a dodecahedron or icosahedron, so I can see that he might have felt no need to include them in his text.

The third body in Figure 7 is a snub cube—a lovely, chiral, tricky-to-cut form. It is discussed in detail in the book, though not named as such, being described simply as "another Geometrical Solid, comprehended under six equal Squares and thirty-two equal equilateral Triangles". In Figure 3, Sharp's drawing is his Fig. 19. Some of the 32 slicing planes for cutting it from a solid cube are indicated in his Fig. 18. This polyhedron had appeared in Albrecht Dürer's *Underweysung der Messung* in 1525 [7], but that is not where Sharp learned of it. In the only morsel of human interest in the entire book, Sharp writes: "The Notion hereof was imparted by a Friend, who understood so much of it as enabled him to draw the several Parts upon Paper or Past-board, and fold them up into a due Form: At his Request I undertook to give a more full and exact Account of all its Parts and Dimensions, and to lay down a regular and certain Method of forming or cutting it." This is notable because calculating the reference points for the slices requires formulating and solving a cubic equation, confirming that Sharp was a creative and able geometer. For a modern derivation, see Lines [18]. Sharp's wormholed model in the Bolling Hall museum may be the oldest snub cube extant.

Guided by Sharp's meticulously calculated dimensions, I have sawn a solid snub cube using a version of Sharp's slicing method. I wonder if I am the first in 300 years to try this. My result is shown in Figure 8. It is a very curious and pleasing object to hold. I tend to continually turn in my hand as if it were necessary to repeatedly verify that one square and four equilateral triangles meet at each vertex. To guide the 32 cuts, instead of marking the cube's surface, I made a custom miter-box. Figure 9 shows the laser-cut plywood parts I prepared; note the slots at the proper angles to hold the saw. Figure 10 shows the assembled box, the cut-off corners, the Japanese pull-saw I used, and the resulting sliced body. The block was rotated after each cut to bring the proper region to the saw slots. The initial cuts removed large corners; later cuts removed smaller remaining bumps. Note that my slicing planes were not precise to 20 decimal places and small errors resulted in faces that are visibly off from equilateral triangles and squares. This was easily adjusted by sanding the surface to produce the final result of Figure 8. I wonder whether Sharp's instrument-making expertise guided him to be more accurate and not require such sanding.

Figure 8. Wood snub cube, made by the author

Figure 9. Laser-cut parts to make miter box to guide slicing

Figure 10. Wood snub cube sliced by the author (before sanding), with custom miter box, saw, and parts removed from the solid cube

Sharp did not use a custom miter box in this way. His method was to draw all the lines on the surface of the block and cut at the lines. Typical woodworking techniques to remove material up to a marked line include sawing, planing, chiseling, whittling with a small knife, and sanding. (A lathe with a spherical chuck might also be used, if starting from a sphere.) Careful examination of Sharp's existing models might indicate his exact cutting methods. A complication is that each cut removes portions of the marked lines that will be needed to guide future cuts. Sharp is specific about how to deal with this: “Always observing to draw all the Lines upon the Cube or Parallelepipedon before any Segment is cut off; and new Lines must be drawn upon the Plane made

by every Section, from the Termination of the Lines which are cut off on every side."

We can be confident that Sharp used this method of redrawing the lines after each cut because he gives a warning that could only come from experience. After explaining where to draw new lines in his simplest example, the tetrahedron, he adds: "But the Neglect hereof, where there are more Bases, will involve the Work in an inextricable Confusion; so that 'tis of absolute Necessity it should be observed in cutting of all the other Bodies, where the Lines and Sections will be more numerous: Wherefore this caution is to be taken along through the whole." It was to avoid the drawing, the redrawing, and the possible confusion that I devised the custom miter-box method.

It is hard to know how Sharp's models would have been perceived in his time: perhaps as elegant educational models, artisanal curios to display, or finely crafted exemplars of a strange mathematical realm. These are the sorts of categorization that would have put them at home in an eighteenth-century cabinet of curiosities. Certainly they wouldn't have been considered fine art or sculpture in any modern sense, as this was long before the idea of non-representational sculpture. Yet I'm certain Sharp enjoyed the process of creating them in the same way that any sculptor must enjoy bringing his or her imaginings to reality. Only a creative passion could lead someone (especially a gentleman of leisure) to invest such focused labor into a block of wood. One can hope that however his models were dispersed, the new owners recognized some of this artistry and kept them safe somewhere. Thus it could happen, over time, that more will come to light.

4. The "Sharpohedron"

I have studied and reproduced all of the polyhedra in *Geometry Improv'd*. With some straightforward graphics programming, a patient coder can translate Sharp's descriptions into computer images or 3D-printed models that are faithful to his geometry. Even though Sharp does not describe his creative intentions, his numbers are so precise that one can test any hypothesis and know for certain whether or not one is thinking of exactly the same shape that he had in mind.

Analyzing all twelve of Sharp's original forms is beyond the scope of this paper, but to do some honor to his vision, we must at least take up one example and see what it is about. Sharp's first and simplest body has eighteen faces. In the 300 years since *Geometry Improv'd* appeared, it has not been re-discovered, reproduced, analyzed, or even mentioned once in the mathematics literature. Yet I find it so worthy that I have dubbed it the "Sharpohedron." Of course, with random slices anyone can create an infinite variety of polyhedra, so why would any particular one be worthy of a name? Indeed, why does "the Mona Lisa," or any human creation warrant a name? Let me try to explain what makes this polyhedron notable and suggest some reasons why, after he imagined it, Sharp might have decided to physically build it and then write about it in such detail for us.

Figure 11. Four versions of Sharpohedron: solid wood, paper, assembled wood, and 3D-printed

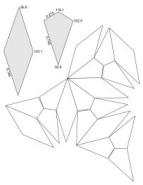


Figure 12. Net and construction data for Sharpohedron (face angles and edge lengths to fit in unit cube). Print net on cardstock, cut outer lines, crease inner lines, and tape together. The dihedral angles are 48.2° (long edges) and 27.3° (short edges).

Figure 11 shows four versions of the Sharpohedron. The first I made from a solid block of basswood following Sharp's precise marking-and-slicing method. The second is made with scissors and paper, by cutting and taping together polygons, following the template of Figure 12. (The tape is hidden on the inside.) The third is made by cutting out eighteen individual faces from plywood, beveling the edges to mate at the proper dihedral angles, and gluing them together. This technique allows for a larger, lighter, hollow wooden model. It is akin to the paper version but stronger, and enriched by the luster and feel of wood. The last is made of ABS plastic by 3D printing [11]. I recommend the reader make some kind of physical model to aid in following the analysis below. The fastest method is to tape together a paper model using the template or data in Figure 12.

With a model in hand, one naturally turns it about to see it from all angles, explores it tactilely to feel its geometry, passes it from hand to hand while trying to understand its structure, and eventually explores the different ways in which it rests on a flat surface. I imagine that Sharp went through such an appreciation process and expected that his followers might come to know each of his twelve original solids from a similar experience with models. A fair test of successful understanding is whether one can close one's eyes and form a mental image clear enough for answering questions like "how many vertices does it have?"

One natural reference point for understanding the Sharpohedron is the regular tetrahedron. As Sharp notes, it "bears some Resemblance to a Pyramid."[6] Like a tetrahedron, the Sharpohedron has four "corners," but these are rounder peaks, less penetrating. While the tetrahedron has three sharply knife-like edges incident to each vertex, here we find that six edges with flatter dihedral angles meet at each peak. The overall form is less of a caltrop and more of an overstuffed tetrahedral pillow, rounded as if it is trying its best to impersonate a sphere. In fact one can imagine a perfect sphere inside it that would be tangent to all eighteen faces. The technical term is that the polyhedron is "circumscribable about a sphere."

Examining the individual faces, one finds two types: six congruent rhombi and twelve congruent kite-shaped quadrilaterals. Kite-shaped faces on polyhedra were extremely rare in the literature before Sharp[7] and might be viewed as implying a kind of experimental artistic daring. Seven of Sharp's new solid bodies involve kites, so he needed a term for them. Because a kite can be assembled by joining half of two different rhombi, he calls them "double Semi-rhombs," then shortens the term to "Semi-rhombs". It is evident that the kites of the Sharpohedron are arranged in four groups of three, with each group positioned like a face of an imagined tetrahedron. The groups are separated from each other by the six rhombi, which are stretched out along the edges of the imagined tetrahedron. Depending on how we turn it, the Sharpohedron may present us with one of its four six-sided peaks or with a Mercedes-like insignia of three kites.

Examination shows that the Sharpohedron's kites and rhombi are not independent shapes. A deeper structure relates them. They share their common long edge length, and the acute angle of the rhombus face is equal to the acute angle of the kite face, so six equal face angles meet at each peak. Also, the short diagonal of the rhombus shape exactly equals the short diagonal of the kite. If we view the kite as "a double semi-rhomb," one of the two rhombus shapes that it derives from is the one we see sitting next to it. Thus each peak is a regular hexagonal pyramid. The overall polyhedron has a 3-fold rotational axis through each vertex, but if you hold it in your hands so just one peak peeks through, what you see has twice as much symmetry: a six-fold rotational axis.

When it is time to place the Sharpohedron down on the table, we discover that it sits in a peculiar manner. With a kite face down, a peak is not quite vertical. This uncooperative personality insists on leaning a bit to the side. Or it can be placed rhombus-down to show a totally different character: Now we feel some tension in the fact that if the lower rhombus points North/South, there is a parallel one on top, but facing East/West. Turning it 45 degrees, so those rhombi point to the NE, SW, NW, and SE, one may suddenly have an Aha! experience of how the form fits snugly into a cubical volume. Of course if you personally sliced it from a wooden cube that will be no shock, but it may be a surprise for those who first understand it as a pillowy tetrahedron. The experience of sliding it in and out of a snug 5-sided cubical box is particularly pleasing.

From Sharp's slicing construction, it is natural to see the Sharpohedron as a special case among a continuum of related forms. First, observe in Figure 13 how one can truncate the vertices of a cube to produce new solids.

There is a continuous range of depths possible and particularly interesting forms arise if one stops either (a) at the depth that leaves regular octagons, giving the Archimedean truncated cube, or (b) at the cube edges' midpoints, giving the cuboctahedron. Or instead of truncating vertices, a related edge-truncation operation produces the twelve slanted squares of the rhombicuboctahedron (Figure 7, left). The Sharpohedron construction can be seen as a variation on these familiar cutting processes.

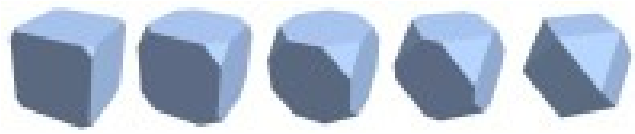


Figure 13. Truncating the vertices of a cube to varying depths.

In the text supporting Sharp's Fig. 1 (our Figure 3), he instructs us to construct the midpoints of the cube's edges and then cut along a plane that goes through one vertex and two midpoints, to remove a corner pyramid like the one shaded in Figure 14. This is done twelve times, once for each cube edge, to produce the twelve kite faces. The six rhombi remain as the central portion of each original cube face, so 12 cuts suffice to make an 18-sided body. (We have seen this before, as it is also how six of the faces of the rhombic triacontahedron arose.) Figure 15 shows one of the marked cube faces; the shaded portions are removed by four cuts, leaving the central rhombus.

Figure 14. Cube shaded to indicate one slanted piece (of twelve) to remove to make a Sharpohedron.

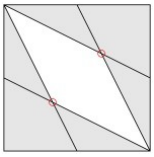


Figure 15. One face of cube marked to indicate cuts.

A very natural generalization of this process is to imagine varying the angle of the cut by constructing two points an arbitrary fraction of the way along the cube edge, instead of exactly half way. Varying the cut depth produces a family of related forms with different kite and rhombus shapes, shown in Figure 16. From this continuum, the one Sharp presented is special not just because the slice goes to the edge midpoints. It is also the only one that is circumscribable about a sphere and that creates six equal face angles and six equal dihedral angles at the points. It makes the two circled intersection points in Figure 15 lie exactly at the one-third and two-third point of the diagonal they lie on, it makes the area of the remaining rhombus exactly one third the area of the square, and it makes the area of each kite exactly one fifth the area of the cube face. There is no way to know which properties of this body Sharp found most interesting, but he did specifically state the circumscribability, the rhombus diagonal (Sqrt[2]/3 for a unit-edge cube), and these surface areas.

Figure 16. Adjusting the depth of the slanted cut through a vertex, varying from a cube to a Sharpohedron to a tetrahedron.

Truncation is just one process for creating new polyhedra from old. Another familiar method is "elevation," i.e., erecting pyramids on the faces of a given polyhedron.[8] Again there is a continuum of possible forms, as we are free to choose the height of the pyramids. Polyhedral elevation has been familiar since 1509 in the images by Leonardo da Vinci for Luca Pacioli's *De Divina Proportione*, in which they elevated polyhedra to a height that created pyramids of equilateral triangles over each face, but there is no record of what Sharp knew. The rhombic dodecahedron can be derived by elevating either the cube or the octahedron; the rhombic triacontahedron can be derived by elevating either the dodecahedron or the icosahedron. In each case, the elevation height is chosen so that pairs of adjacent isosceles triangles become co-planar and merge across an edge to become rhombi. Sharp discusses these relationships without specific mention of elevation or pyramids, so it is not clear if he thought of elevation as a formalized conceptual process. One of his later bodies (consisting of 24 isosceles triangles) is an elevated cube, which he describes as being "compounded of six square Pyramids ... and a Cube." See his Fig. 24, parts 1 and 2, in Figure 3. Another (consisting of sixty isosceles triangles) can be derived by elevating the dodecahedron and yet another (consisting of eighty triangles) can be derived by elevating the pentagons of an icosidodecahedron. See his Figs. 34 and 38 in Figure 6. In these three cases, the elevation height puts all vertices on a common circumsphere. But Sharp gives us just the results so we don't know how he came to think of them.

Figure 17. Starting with a truncated tetrahedron, elevating first the hexagons, then the triangles, to create Sharpohedron.

With this background, we note that the Sharpohedron can be simply derived by elevating the Archimedean truncated tetrahedron. Figure 17 illustrates the process. The truncated tetrahedron comprises four regular hexagons and four equilateral triangles. We can first elevate the hexagons into pyramids of a height where three of their isosceles triangles merge with neighbors to become the rhombi, and then elevate the triangles until their isosceles triangles merge with the remaining faces of the hexagonal pyramids to become kites. This is another elegant way to understand the Sharpohedron. If you pick up your model and visualize just the short diagonals of the faces, you can see the truncated tetrahedron hiding within it. This construction explains why the overall shape has three-fold rotational symmetry, yet the peaks are regular six-sided pyramids. We can also understand why there are eighteen faces: because the truncated tetrahedron has eighteen edges. Once elevation is understood, it seems natural to apply it to familiar Archimedean solids, but there is no way to know whether Sharp thought along these lines. He only tells us the final shape.

A different way to specify a polyhedron is to list a finite set of infinite bounding planes, with the understanding that we are interested in the region of 3D space that is interior to all the planes. For example, we can give equations for six planes ($x=1$, $x=-1$, $y=1$, $y=-1$, $z=1$, $z=-1$) to specify the cube that lies interior to them.[9] Abraham Sharp appears to be thinking very much along these lines when he defines cut planes and discards all the exterior regions. For construction purposes, he is very specific in locating points on the surface of the starting block that delimit how the plane intersects the surface, but he also understands more abstractly that any way of characterizing planes determines a unique result. For example, he explains how the rhombic dodecahedron is "apparently deriv'd either from the Cube or Octahedron, by fixing Planes upon every Edge perpendicular to that Axis which passeth from the Center of the Body through the middle of each Edge..." He similarly finds the face planes of a rhombic triacontahedron as planes through the edge midpoints of a dodecahedron or icosahedron.

With this type of characterization in mind, one might ask what happens with other "natural" sets of planes, for example ones derived from the edge midpoints of other simple polyhedra. (Given a point, a uniquely defined plane to choose, as Sharp notes, is the one that passes through the point and is orthogonal to the line that connects the point to the center of the body.) Once this process is familiar, it turns out that the Sharpohedron can be beautifully and elegantly derived as what is inside the planes defined by the edge midpoints of a truncated tetrahedron. This is my best guess as to how Sharp derived it, simply applying an operation he understood on Platonic solids to the simplest Archimedean solid, but there is no way to know for sure. The conjecture is supported by the fact that *Geometry Improv'd* also includes a body with 36 faces, which can be derived

analogously from the edge midpoints of a truncated octahedron. See Fig. 30, parts 1 and 2, in Figure 3. However, others of his twelve original bodies have separate derivations. Sharp's second body, one with 24 faces, can be derived by applying this process not to the edge midpoints, but to the 24 vertices of a rhombicuboctahedron. Here Sharp appears to be anticipating the idea behind the Catalan solids, which arise if one defines a plane for each vertex of an Archimedean solid. But he was not consistent in this method; some of his other bodies are approximate Catalan solids with slightly different dimensions.

So there are a number of possible routes by which Sharp might have arrived at his 18-sided body (just as networks of interrelationships arise in any interesting mathematical domain). And there is a variety of distinctive geometric and personality characteristics that could have bolstered his assessment of the Sharpohedron's beauty and elegance. We will never know for sure how Sharp felt about his twelve "children" that he presented in *Geometry Improv'd*, but there is strong evidence that the Sharpohedron was particularly important to him. An indication of its significance can be found in Vertue's posthumous engraving of Sharp. The geometric figure that appears on the panel within the picture is a portrait of the Sharpohedron, centered on a group of three kite faces and perhaps slightly atilt, as it tends to be. It is unclear how Vertue knew of the Sharpohedron, but for some reason he chose this particular example to embody Sharp's mathematical work in the 1744 portrait. This was just two years after Sharp's death, so he would have been able to learn from those who knew Sharp which form was most significant.

5. Conclusion: Sharp the Artist

Abraham Sharp's self-taught geometric genius appears to have come out of nowhere, gone in its own direction, and disappeared, not to be pursued for 300 years. Although biographies label him as a mathematician (or more precisely "an ingenious mathematician, mechanist, and astronomer," per Hutton), I categorize him first as an under-appreciated artist. Yes, he certainly had the technical skills for calculation and could apply mathematics to solve difficult problems, but there was much more to his creative compulsion. Mathematics was just one of the abilities, along with mechanical skill, craftsmanship, perseverance, and imagination, that he applied toward his artistic aims.

Of course the term "art" has had differing meanings and associations over the centuries and one can not claim that Sharp himself saw his polyhedra project as fine art in any modern sense. But certainly he gave us what modern definitions call for: artifacts that express the creator's imagination, conceptual ideas, or technical skill, intended to be appreciated primarily for their beauty or emotional power. The beauty Sharp saw in his creations undoubtedly included both a visual layer, as one might enjoy the gleaming craftsmanship of a brass scientific instrument, and a deeper mathematical layer, as when one understands how the relationships of truncation, elevation, and other geometric transformations provide a mental graph connecting different structures.

While it is always risky to guess at someone's intentions, especially one living in times so different from ours, we do have Sharp's statements that he was interested in creating attractive new works for viewers to appreciate. At the point in *Geometry Improv'd* where he introduces his novel polyhedra, he says: "as an Addition to the Geometrical Store, I shall subjoin Twelve more; none of which (I presume) have yet been expos'd to publick View, and some of them perhaps being more beautiful and elegant than any of the former." He wants to share his creations, and his exacting calculations might be seen as the extreme perfectionism of an artist who wishes his work to be "just so."

Mathematicians have ignored *Geometry Improv'd* precisely because it is more art than math. They look to books for logical development and conceptual understanding, not for lists of idiosyncratic instructions. Lacking a definition-theorem-proof arrangement, Sharp's writing does not immediately show a casual reader that he indeed thinks like a mathematician. The underlying structures and relationships that connect his creations to each other and to the existing body of geometric knowledge are not spelled out. A leap of faith is required to invest the time needed to dive in and appreciate his unique contributions.

Sharp worked meticulously and at length both on the calculations and on the woodworking that resulted in his

physical models. He saw the visual and conceptual value in his geometric creations and freely presented them to the world, for us to appreciate both in print and as concrete objects. The very human pleasure of visualization must have underlain all this. Both as an instrument maker and as a polyhedron slicer, Sharp would begin with a concept in his mind's eye, one that he felt was worthy of physical existence, and take whatever steps were required to bring it to reality. This is the creative drive which is the engine inside any artist.

The individual character of a work may place it somewhere along a continuum between mathematical creation and fine art. Many beautiful mathematical structures seem to have a certain inevitability and it happens that different mathematicians formulate them independently. (This is why some feel mathematics is *discovered* in a Platonic realm rather than *invented*.) At the other extreme, unequivocal works of fine art typically show the hand of one particular artist. We feel that if Beethoven had never lived, no one else would ever have produced any particular piano sonata of his. Within this spectrum, where can we can place Sharp's polyhedra? I see the fact that no one else came up with his Mona Lisa, the Sharpohedron, in over 300 years as a testament to Sharp's personal artistry, but it is the presentation in *Geometry Improv'd* that is absolutely unique. The hand of the artist is evident throughout the book in every square inch of the densely packed engravings and in every 20-digit measurement.

By thinking of Sharp as an artist and his polyhedra project as art, we can make some sense of the enormous number of digits he presents. The unnecessary over-precision of the dimensions might incorrectly suggest (to a reader unfamiliar with his instrument-making fame) that this was the writing of a very abstract mathematician, one out of touch with the practicalities of real-world fabrication. No metal-smith could work to even four digits of precision, so why waste time doing twenty-digit calculations and why waste space in the book printing such exact values? One might propose that he was merely showing off his mastery of calculation, of trigonometry, and of logarithms, but that doesn't accord with his modest nature. I have come to view his over-precision as a form of artistic style. The digits are a kind of decoration he adopted, a brush stroke of sorts, akin to Van Gogh's swirls, Seurat's dots, or Beethoven's tremolos. Sharp is decorating his document with digits in an expression of personal taste, thereby creating a unique impression on the reader.

I see an (admittedly anachronistic) analogy in all this to Sol Lewitt's twentieth-century conceptual artwork. Lewitt emphasized the role of the artist as a thinker rather than craftsperson and gave constructive instructions for wall drawings to be painstakingly executed by assistants, allowing for them to be reproduced over time in different locations. He thereby separated the essence of a visual artwork from its repeatable process, just as a composer's musical score provides a representation that is of a distinct nature from its many possible performances. In an analogous manner, Sharp's presentation is very much an instruction sheet that merely commands us in how to proceed mechanically. Lewitt was happy for different practitioners to adapt his wall drawing instructions to their own particular rooms, wall sizes, wall materials, etc. Similarly, I believe Sharp would be pleased to see his creations rediscovered and newly instantiated not just in boxwood, but in 3D printing, computer graphics, and any other medium yet to be invented.

It took some decades for Lewitt's notion of conceptual art to be broadly understood by the public, but now one finds giant wall drawings of his at large museums all over the world. Similarly, many works of Beethoven and other innovative artists were not immediately popular, but came to be understood over time. Perhaps after 300 years, the moment has come for a wider appreciation of Abraham Sharp's remarkable artistry. It would be wonderful to see the polyhedra of this multifaceted genius wrought large. A wider recognition of his uniquely mathematical artwork would certainly be deserv'd.

Acknowledgment
I thank the many curators and collections managers who generously researched their holdings at my request, especially Gearóid Mac a' Ghobhainn at the Bradford Museum and Galleries, Graham Dolan at the Royal Observatory, Greenwich, Jonathan Bushell at the Royal Society, Oliver House at the Bodleian Libraries, Joshua Nall at the Whipple Museum, Cambridge, David Glover of the Halifax Antiquarian Society, Frances Chambers at the Yorkshire Philosophical Society, Helen Rawson at York Minster, and Katie McAdam at the Calderdale Museums.

Footnotes

1. The rhombic dodecahedron, rhombic triacontahedron, and Archimedean solids were described in Kepler's 1619 *Harmonices Mundi* [17], which Sharp could have encountered in his astronomical work, but he does not specify any sources other than Euclid.

2. Sharp also begins with a sphere in some of his instructions, and as a scientific instrument maker was an expert using and making lathes, but nothing in his book discusses using a lathe to make polyhedra.

3. It is not that Sharp was a "pure" mathematician who eschewed all applications. He was very applied in his instrument making and seems enthusiastic in describing how the faces of a snub cube could be used as the planes for a set of sundials. Nowadays, a utilitarian might propose using new polyhedra for dice.

4. Starting with a block the size of the Earth, twenty decimal places gives a resolution smaller than an atom. I have spot-checked a few dozen of Sharp's calculations and found he is usually correct to as many digits as he lists, though he is occasionally incorrect in the least significant three to five digits. For example, one of the distances to mark off when constructing the icosahedron is (1/2)Sqrt[15]-Sqrt[3], which he gives as 0.20444086553483114889 23 but the correct value is 0.20444086553483114906 22.

5. I learned of the book from the epigraph and a footnote in the Polyhedra chapter by Coxeter in [2]. But even Coxeter doesn't delve into the technical details, merely commenting how Sharp's figure with 90 faces "somewhat resembles" a later polyhedron discovery, the rhombic enneacontahedron. (I have checked that it is topologically equivalent, but geometrically distinct.) The only other reference I know in the mathematics literature is in a discussion of Descartes' understanding of polyhedra [8], where the modern editor cites *Geometry Improv'd* as a general indication of the contemporary state of the art. In the world of fine art, Raphaël Zarka references Sharp in a sculpture series consisting of wood beams marked with burned lines [24]. An interesting paper from the Oxford Master's program in Literature and Arts takes an interdisciplinary look at Vertue's work and is the only printed reference I know that specifically discusses any of Sharp's individual polyhedra, the "Solid of Eighteen Bases" [20].

6. A few comments on symmetry are appropriate here, as it is central to any modern treatment of polyhedra. The Sharpohedron has "tetrahedral symmetry with reflections," i.e., what is now called orbifold type *332. This means that it has four axes of three-fold rotational symmetry, three axes of two-fold rotational symmetry, and six mirror planes. Sharp worked long before the nineteenth century development of group theory formalized symmetry, so he would not see it in these terms, but he certainly had an adequate intuitive notion. All twelve of the bodies he created have some sort of polyhedral point group symmetry, with the Sharpohedron standing alone as his only one in the *332 class. (His later ones fall into four other groups: octahedral or icosahedral rotations, with or without reflections.) In his instructions, Sharp typically details a half-dozen cutting planes, then says "and more like these," relying on the reader's intuitive understanding of the correct symmetry for each example. When coding up his slicing process, I found it convenient to specify just one cutting plane, and then generate the others algorithmically by applying the appropriate group of symmetry transformations. If Sharp had formalized an analogous notion, his book could have been even shorter.

7. I am aware of only one previously published polyhedron with kite-shaped faces, which is found among Wentzel Jamnitzer's 1658 imaginings [16]. In later centuries they became commonplace with the Catalan solids.

8. Some people have misunderstood the term "stellation" and used it for this, but I prefer to use "stellation" as Kepler originally defined it—for a process of extending the face planes [17].

9. In the twentieth-century formalization of this idea as "Nef polyhedra," one constructs the intersection of a finite set of "half-spaces" defined by the planes. This method can only produce convex polyhedra, but it is suitable because all of Sharp's examples are convex.

References

[1] ArtUK, Portrait of Abraham Sharp, https://artuk.org/discover/artworks/abraham-sharp-16511742-23612

[2] W.W. Rouse Ball and H.S.M. Coxeter, *Mathematical Recreations and Essays,* Dover, 1987.

[3] Whiston Bristow, *A Catalogue of the Genuine and Valuable Collection of that Well Known Antiquarian the Late Ralph Thoresby, ... All which Will be Sold by Auction,* 1764, https://books.google.ca/books?id=XnpdAAAAcAAJ

[4] M.E. Catalan, "Mémoire sur la théorie des polyèdres," Journal de l'École Polytechnique, 1865, pp. 1-71.

[5] Elizabeth Connor, "Abraham Sharp, 1653-1742," Publications of the Astronomical Society of the Pacific, V. 54, No. 321, 1942, pp. 237-243.

[6] William Cudworth, *Life and Correspondence of Abraham Sharp: The Yorkshire Mathematician and Astronomer, and Assistant of Flamsteed; with Memorials of His Family, and Associated Families,* S. Low, Marston, Searle & Rivington, Ltd., 1889. https://books.google.ca/books/about/Life_and_Correspondence_of_Abraham_Sharp.html?id=yhMzAQAAMAAJ&redir_esc=y

[7] Albrecht Dürer, *Unterweysung Messung,* 1525, English translation as *Painter's Manual,* edited by Walter Strauss, Abaris Books, 1977.

[8] P.J. Federico, *Descartes on Polyhedra: A Study of De Solidorum Elementis,* Springer, 1982.

[9] E. Forbes et al., eds., *The Correspondence of John Flamsteed,* CRC Press, 1995, Vol 2, pp. 908-916.

[10] George Hart, Abraham Sharp's Polyhedra, http://georgehart.com/virtual-polyhedra/sharp.html

[11] George Hart, Abraham Sharp's Geometry Improv'd, http://georgehart.com/sharp

[12] N. S. Heineken, "Relics of the Mechanical Productions of Abraham Sharp, the Assistant of Flamsteed," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science,* Series 3, Vol. 30, 1847, pp. 25-27.

[13] John Jacob Holtzapffel, *Turning and Mechanical Manipulation,* Holtzapffel & Co., London, 1879-1884.

[14] Henry L.P. Hulbert, *Sir Francis Sharp Powell, Baronet and Member of Parlament,* Jackson, 1914.

[15] Charles Hutton, *A Philosophical and Mathematical Dictionary,* London, 1795, Vol. 2, pp. 385-387. https://books.google.ca/books/about/A_Mathematical_and_Philosophical_Diction.html?id=Nn9RAAAAYAAJ&redir_esc=y

[16] Wentzel Jamnitzer, *Perspectiva Corporum Regularium,* 1568.

[17] Johannes Kepler, *Harmonices Mundi*, 1619.

[18] L. Lines, *Solid Geometry,* Dover, 1965.

[19] Abraham Sharp, *Geometry Improv'd,* London: Richard Mount and John Sprint, 1717, https://www.e-rara.ch/zut/content/structure/1329239

[20] Tyson Rallens, "Proof and Possession: A Conversation about Knowledge-Making between Two Artifacts of the 1740s," VIDES, 2014, pp. 173—184, https://open.conted.ox.ac.uk/series/mla-vides-2014

[21] Ralph Thoresby, *Ducatus Leodiensis,* including a catalog of the Musaeum Thoresbyanum, 1713, http://drc.usask.ca/projects/ark/public/catalog.php?catalog=ThorMus

[22] George Vertue, Engraving of Abraham Sharp, 1744, Oxford History of Science Museum, http://www.mhs.ox.ac.uk/collections/imu-search-page/record-details/?thumbnails=on&irn=10916&TitInventoryNo=79701

[23] Frances Willmoth, "Sharp, Abraham," *Oxford Dictionary of National Biography,* Online Edition, Oxford University Press, 2004, http://www.oxforddnb.com/view/article/25206

[24] Raphaël Zarka, “Billes de Sharp,” Kadist Gallery, 2008, https://kadist.org/work/billes-de-sharp-n578/

Figure Captions

1. Title page of *Geometry Improv'd*
2. Sharp's text showing exact values and decimal approximations (up to 28 digits) of dimensions, surface area, and volume of one of his original solids.
3. Plate I of *Geometry Improv'd*
4. Wood Rhombic Triacontahedron, sliced by the author
5. Engraved Portrait of Abraham Sharp by George Vertue
6. Plate II of *Geometry Improv'd*
7. Three wood polyhedra made by Abraham Sharp: rhombicuboctahedron, icosidodecahedron, and snub cube. Image courtesy of Bradford Museums and Galleries.
8. Wood snub cube, made by the author
9. Laser-cut parts to make miter box to guide slicing
10. Wood snub cube sliced by the author (before sanding), with custom miter box, saw, and parts removed from the solid cube
11. Four versions of Sharpohedron: solid wood, paper, assembled wood, and 3D-printed
12. Net and construction data for Sharpohedron (face angles and edge lengths to fit in unit cube). Print net on cardstock, cut outer lines, crease inner lines, and tape together. The dihedral angles are 48.2° (long edges) and 27.3° (short edges).
13. Truncating the vertices of a cube to varying depths.
14. Cube shaded to indicate one slanted piece (of twelve) to remove to make a Sharpohedron.
15. One face of cube marked to indicate cuts.
16. Adjusting the depth of the slanted cut through a vertex, varying from a cube to a Sharpohedron to a tetrahedron.
17. Starting with a truncated tetrahedron, elevating first the hexagons, then the triangles, to create Sharpohedron.

Note to readers: *This pdf includes only low-resolution placeholders for the images. High resolution figures can be downloaded from:* `http://georgehart.com/private/Sharp-figures.zip`